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CERTAIN FINITE DIMENSIONAL MAPS AND THEIR APPLICATION TO HYPERSPACES

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ABSTRACT

In [8] Y. Sternfeld and this author gave a positive answer to the following longstanding open problem: Is the hyperspace (= the space of all subcontinua endowed with the Hausdorff metric) of a 2-dimensional continuum infinite dimensional? This result was improved in [9] where it was shown that for every positive integer number n a 2-dimensional continuum contains a 1-dimensional subcontiuum with hyperspace of dimension $\geq n$. And it was asked there: Does a 2-dimensional continuum contain a 1-dimensional subcontinuum with infinite dimensional hyperspace? In this note we answer this question in the positive.

Our proof applies maps with the following properties. A real valued map f on a compactum X is called a Bing map if every continuum that is contained in a fiber of f is hereditarily indecomposable. f is called an n-dimensional Lelek map if the union of all non-trivial continua which are contained in the fibers of f is n-dimensional. It is shown that for $\dim X = n+1$ the maps which are both Bing and n-dimensional Lelek maps form a dense G_δ-subset of the function space $C(X, \mathbb{I})$.

1. Introduction

All spaces are assumed to be separable metrizable. $\mathbb{I} = [0,1]$. By a map we mean a continuous function.

Let X be a compactum (=compact space). 2^X denotes the space of closed subsets of X endowed with the Hausdorff metric, and $\mathcal{C}(X)$ is the subset of 2^X

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which consists of the subcontinua (= compact connected subsets) of X. Both 2^X and $\mathcal{C}(X)$ are compact.

In [8] Y. Sternfeld and this author proved that if $\dim X = 2$ then $\dim \mathcal{C}(X) = \infty$. In [9] this result was improved by showing that actually the 1-dimensional subcontinua of X are responsible for the infinite dimensionality of $\mathcal{C}(X)$, more precisely: for every positive integer n, X contains a one-dimensional subcontinuum T_n with $\dim \mathcal{C}(T_n) \geq n$ and it was asked: Does there exist a 1-dimensional subcontinuum T with $\dim \mathcal{C}(T) = \infty$? In this note we give a positive answer to this question:

THEOREM 1.1: *Let X be a 2-dimensional continuum. X contains a 1-dimensional subcontinuum T with* $\dim \mathcal{C}(T) = \infty$.

Our approach develops the one of [6] and [9] and applies finite dimensional maps with certain properties.

A continuum is indecomposable if it is not the sum of two proper subcontinua. It is hereditarily indecomposable if each of its subcontinua is indecomposable. A compactum is said to be a Bing compactum if its subcontinua are hereditarily indecomposable and a map is said to be a Bing map if its fibers are Bing compacta. In [7] the following was proved:

THEOREM 1.2: *Let X be a compactum. Almost all maps in the function space $C(X, \mathbb{I})$ are Bing maps (where almost all= all but a set of first category).*

(See [3] for another approach to constructing Bing maps.)

A map f on a compactum is said to be an n-dimensional Lelek map if the union of all non-trivial continua contained in the fibers of f is of dimension $\leq n$ (Lelek [5] constructed such an n-dimensional map from $\mathbb{I}^{n+1}$ onto a dendrite, see also [11]). In Section 2 we study Lelek maps and prove:

THEOREM 1.3: *Let X be an $(n+1)$-dimensional compactum. Almost all maps in the function space $C(X, \mathbb{I})$ are n-dimensional Lelek maps.*

In Section 3, Lelek and Bing maps are applied to prove Theorem 1.1.

2. Lelek maps

Let $f\colon X \longrightarrow Y$. For $a > 0$ denote

$F(f, a)$ = the union of components A of the fibers of f with $\operatorname{diam} A \geq a$, and $F(f) = \bigcup_{i=1,\infty} F(f, 1/i)$.

Clearly $F(f, a)$ is closed and hence $F(f)$ is σ-compact. f is an n-dimensional Lelek map if $\dim F(f) \leq n$.

PROPOSITION 2.1: *Let X and Y be compacta. For every $n > 0$, the n-dimensional Lelek maps form a G_δ-subset of $C(X, Y)$.*

Proof: Let $a, b > 0$. Denote $U(a,b) = \{f \in C(X,Y)\colon d_{n+1}(F(f,a)) < b\}$ where $d_{n+1}(F) < b$ if there exists an open cover of F with mesh $< b$ and order$\leq n$ (see [4], p. 105). It is easy to verify that $U(a,b)$ is open in $C(X,Y)$. Then the set of n-dimensional Lelek maps $= \bigcap_{i,j=1,\infty} U(1/i, 1/j)$ and we are done. ∎

PROPOSITION 2.2: *Let X be an $(n+1)$-dimensional compactum. The set of n-dimensional Lelek maps is dense in $C(X, \mathbb{I})$.*

We give two proofs of this proposition. The first proof (A) is based on [11]. The second one (B) is selfcontained.

Proof A: Let $\mathbb{I} \subset D$ be embedded in a dendrite D with a dense set of end points and let $r\colon D \longrightarrow \mathbb{I}$ be a 0-dimensional retraction. r induces the continuous surjection $r^*\colon C(X,D) \longrightarrow C(X,\mathbb{I})$ defined by $r^*(f) = r \circ f$. Since r is 0-dimensional, f and $r \circ f$ have the same collection of non-trivial continua which are contained in their fibers. Hence r^* maps n-dimensional Lelek maps to n-dimensional Lelek maps. Take a 0-dimensional σ-compact subset Z of X such that $\dim X \setminus Z \leq n$. By Theorem 1.1 of [11] for almost all maps f in $C(X, D)$ we have $\{x\} = f^{-1}f(x)$ for every $x \in Z$. This implies that such f are n-dimensional Lelek maps and by the above $r \circ f$ form a dense subset of n-dimensional Lelek maps in $C(X,\mathbb{I})$. ∎

Proof B: Let Z be 0-dimensional and closed in X. For $a > 0$ denote

$$H(Z,a) = \{f \in C(X,\mathbb{I})\colon F(f,a) \cap Z = \emptyset\}.$$

It is clear that $H(Z,a)$ is open in $C(X,\mathbb{I})$ and let us show that $H(Z,a)$ is dense in $C(X,\mathbb{I})$.

Let $\epsilon > 0$ and let $f\colon X \longrightarrow \mathbb{I}$. We are going to ϵ-approximate f by a $g \in H(Z,a)$. Take a finite open cover $V_1, V_2, \ldots, V_k$ of Z such that $\mathrm{cl}V_i$ are disjoint, $\mathrm{diam}\ \mathrm{cl}V_i < a$ and $\mathrm{diam}\ f(\mathrm{cl}V_i) < \epsilon$. Pick $g_i\colon \mathrm{cl}V_i \longrightarrow \mathbb{I}$ such that g_i are ϵ-close to f on $\mathrm{cl}V_i$ and

(*) $g_i(\partial V_i)$ are singletons and $g_i^{-1}(g_i(\partial V_i)) = \partial V_i$.

Now extend g_i to a $g \in C(X,\mathbb{I})$ ϵ-close to f and let us show that $g \in H(Z,a)$. Indeed, let a continuum T with $\mathrm{diam}T \geq a$ be contained in a fiber of g. T does not intersect V_i since else T would intersect ∂V_i and hence would not meet V_i by (*). So $g \in H(Z,a)$ and this proves that $H(Z,a)$ is dense.

Take 0-dimensional compact subsets Z_i of X such that $\dim X \setminus \bigcup Z_i \leq n$. Then by the above $H = \bigcap_{i,j} H(Z_i, 1/j)$ is dense in $C(X, \mathbb{I})$ and for every $f \in H$, $F(f)$ does not meet Z_i and hence $F(f) \subset X \setminus \bigcup Z_i$. Thus $\dim F(f) \leq n$ and we are done. ∎

Propositions 2.1 and 2.2 imply Theorem 1.3.

Note that Proposition 2.2 and hence Theorem 1.3 hold for $(n+1-k)$-dimensional Lelek maps to $\mathbb{I}^k$.

Indeed, let $f = (f_1, \dots, f_k)$: $X \longrightarrow \mathbb{I}^k$. Approximate f_1 by an n-dimensional Lelek map f_1^*, i.e. $\dim F(f_1^*) = n$. Since $F(f_1^*)$ is σ-compact, by Theorem 1.3 f_2 can be approximated by a map f_2^* such that $\dim F(f_2^*|_{F(f_1^*)}) \leq n-1$. Then (f_1^*, f_2^*) is an $(n-1)$-dimensional Lelek map and continue by induction on k.

3. Hyperspaces of 2-dimensional continua

The main aim of this section is to prove Theorem 1.1. We will prove:

THEOREM 3.1: *Every 2-dimensional compactum X contains a 1-dimensional subcontinuum K which admits an open monotone map q: $K \longrightarrow T$ onto a continuum T of* $\dim \geq 2$.

Theorem 1.1 can easily be derived from Theorem 3.1: by [8] $\mathcal{C}(T)$ is infinite dimensional and since q is open and monotone on K, $\mathcal{C}(T)$ can be embedded in $\mathcal{C}(K)$ by $D \in \mathcal{C}(T) \longrightarrow q^{-1}(D) \in \mathcal{C}(K)$ and hence $\mathcal{C}(K)$ is also infinite dimensional.

Proof of Theorem 3.1: A map $W : \mathcal{C}(X) \longrightarrow \mathbb{R}^+$ is called a Whitney map if W vanishes on the set of singletons in $\mathcal{C}(X)$ and if $A \neq B$ in $\mathcal{C}(X)$ and $A \subset B$ implies $W(A) < W(B)$ (see [10] for more information on hyperspaces).

Whitney maps always exist: if $\{f_n\}_{n=1}^{\infty}$ is a dense sequence of functions in $C(X, \mathbb{I})$ and $W_n(A) = \operatorname{diam} f_n(A)$, W_n: $\mathcal{C}(X) \longrightarrow \mathbb{I}$ then $W = \sum_{n=1}^{\infty} W_n/2^n$ is a Whitney map.

Let W be a Whitney map for X. Take a 1-dimensional Bing–Lelek map $f \in C(X, \mathbb{I})$, i.e. a map which is both a Bing map and a 1-dimensional Lelek map. Such a map exists by Theorems 1.2 and 1.3. Let $f = h \circ g$ be the monotone-light decomposition of f with g monotone and h light. Then g is a 1-dimensional Bing–Lelek map to the 1-dimensional compactum $Z = g(X)$.

Now we will apply a construction from [6]: for $a > 0$ define a decomposition $\mathcal{A}$ of X by: the fibers $g^{-1}(z)$ of g with $W(g^{-1}(z)) < a$ and the continua A with $W(A) = a$ which are contained in the fibers $g^{-1}(z)$ of g with $W(g^{-1}(z)) \geq a$.

It is easy to check that $\mathcal{A}$ covers X. To show that $\mathcal{A}$ is a decomposition we have to show that every 2 elements $A \neq B$ of $\mathcal{A}$ which are contained in a fiber $g^{-1}(z)$ with $W(g^{-1}(z)) \geq a$ are disjoint. Each of them cannot be contained in the other as $W(A) = W(B)$ and since $g^{-1}(z)$ is hereditarily indecomposable they have to be disjoint.

(i) *$\mathcal{A}$ is upper semicontinuous.*

Proof: Let $A_1 \in g^{-1}(z_1), A_2 \in g^{-1}(z_2), \ldots$ be a sequence of elements of $\mathcal{A}$ such that $\lim A_i = A$ in 2^X. Then $z = g(A) = \lim z_i$ and $W(A) = \lim W(A_i) \leq a$. It is easy to see that the two last conditions imply that A is contained in an element of $\mathcal{A}$ and (i) follows.

Let q: $X \longrightarrow q(X)$ be the quotient map associated with $\mathcal{A}$. Clearly q is monotone. Denote $F = \bigcup\{g^{-1}(z)\colon W(g^{-1}(z)) \geq a\}$. $F = q^{-1}(q(F))$ and F is closed and 1-dimensional since g is a 1-dimensional Lelek map.

(ii) *The restriction of q to F is an open map.*

Proof: Let $A_1, A_2, \ldots$ be a sequence of elements of $\mathcal{A}$ which are contained in F, that is $W(A_i) = a$, and let $\lim A_i = A$ in 2^X. Then $W(A) = a$ and A is contained in a fiber of g since A_i are contained in fibers of g. Hence $A \in \mathcal{A}$ and $A \subset F$ and (ii) follows.

Clearly $W(A) \leq a$ for $A \in \mathcal{A}$. So we can assume that for a small enough mesh$\mathcal{A}$ will be so small that $\dim q(X) \geq \dim X = 2$ (see Corollary 9, p. 111, [4]). q coincides with g on $X \setminus F$ and hence $\dim q(X \setminus F) = \dim g(X \setminus F) \leq \dim g(X) = 1$. Then since F is closed $\dim q(X) = \max\{\dim q(F), \dim q(X \setminus F)\}$ and therefore $\dim q(F) \geq 2$.

Now pick a subcontinuum T of $q(F)$ with $\dim T \geq 2$ and define $K = q^{-1}(T)$. K is a continuum as q is monotone and K is 1-dimensional since $K \subset F$. The restriction of q to K is an open map by (ii) and we are done. ∎

Note that for a finite dimensional X with $\dim X \geq 3$, Theorem 3.1 holds with T infinite dimensional. Indeed, by [1] every compactum of dim ≥ 3 contains a hereditarily indecomposable subcontinuum of dim ≥ 2 which by Theorem 1.2 of [6] contains a 1-dimensional subcontinuum that admits an open monotone map onto an infinite dimensional continuum.

On the other hand, one can check that from Moore's theorem ([12], p. 171, Theorem 2.1′) it follows that the monotone image of a closed subset of a 2-dimensional sphere is at most 2-dimensional. This shows that the estimate of the dimension of T in Theorem 3.1 cannot be improved.

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